

Filtering theory with orthogonal functions for two-dimensional astrometry

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Received April 27, 1994; accepted March 18, 1995

Abstract. — A new technique of numerical filters to evaluate the brightness, position of photocenter and other properties of celestial objects from their two-dimensional CCD images is presented. The filters are based on a set of two-dimensional orthogonal functions, and an appropriate linear combination of the orthogonal functions enables a filter to achieve the minimum variance estimation. Theoretical properties of our filters (called the ORTHO filters) are compared with those of some other methods or techniques proposed so far for one- or two-dimensional image location problems. The results of numerical simulations are also presented for a verification of the theoretical expectations.

Key words: astrometry — methods: data analysis

1. Introduction

The advent of CCDs has brought several important changes in the mode of astronomical observations and their reduction. For example, in the field of astrometry it is now becoming possible with the use of a CCD meridian circle to make observations down to 16th to 18th magnitude (Yoshizawa et al. 1993; Stone 1993). This situation requires us to develop an efficient method for the determination of the precise locations of millions of stars which we will have after observing a part of the sky with a CCD meridian circle. The present paper is concerned with the practice of the two-dimensional image location problems to be encountered with reduction of CCD images in general.

An image location problem is a branch of non-linear parameter estimations. A scheme (or a method) to find the value of a location parameter, e.g. the location of a star image, is called a location estimator. The basic characteristics of several popular (one-dimensional and two-dimensional) location estimators, say, the Maximum Likelihood (ML) method, the Least Squares (LSQ) method and so on, are given in many references (cf. Lindgren 1978; Yoshizawa et al. 1985; Stone 1989). It is well known that the exact ML method is the optimal estimator for the case of non-Gaussian error distributions. [In the case of Gaussian error distribution, the LSQ method is the ML method of known variance.] The ML method requires, however, in general so much computing time because of its strong nonlinearity (or iteration process) that the direct application of the exact ML method to treat millions of

stars is not easy in practice. The ML method proposed by Lu (1993) is, indeed, the ML method *not* for Poissonian process but for Gaussian process according to Eq. (2) in his paper.

A numerical filter method is a simplified scheme to solve the exact problem for a given estimator. It is clear from the work by Yoshizawa et al. (1985) that the degradation of the method due to the use of the simplified scheme is small, about 5% increase in the standard deviation, whereas the reduction of the computing time may be an order of magnitude at least. Thus the adoption of the numerical filter method is one of our guiding principles in developing a new efficient two-dimensional numerical filter estimator of image location.

Another important factor to be considered is the applicability of the developed method to various kinds of astronomical objects, not only to an axi-symmetric Gaussian image but also to a multiple or non-symmetric distribution of light, say like a galaxy. To this end we develop in Sect. 2 a two-dimensional numerical filter method by using a series of orthogonal functions which satisfy the normalization conditions with Gaussian weight. The filters so developed are called ORTHO filters. In Sect. 3 we examine the performance of our (orthogonal) numerical filters to find the location of a symmetric Gaussian image, and shall compare it with that of other methods. The results of some numerical simulations are given in Sect. 4 to confirm our theoretical expectations.

2. Construction of orthogonal series of filter estimators

2.1. Basic concepts

A stellar image in the telescope focal plane is approximated by a Gaussian distribution because of Airy diffraction, atmospheric turbulence, etc. When a stellar image taken with a CCD array is modeled by means of an axisymmetric Gaussian with Poissonian noise process, the estimation of the center of the Gaussian image is optimally made by the ML location estimator.

The arithmetic mean is equivalent to the ML estimator when there is no background contribution, therefore it is also optimal (Appendix A.1). However, when the contribution of the sky brightness and the dark counts are not small, the arithmetic mean is no longer the optimal estimator. Since noise that originates far from the image center degrades the estimator, it is therefore better to reduce the weights far from the image center. In fact, the shape of the ML filters looks like the arithmetic mean filter convolved with a gradual cut off as illustrated in Fig. 1 for the one-dimensional case.

In the ORTHO filters developed below, the reduction in weight is obtained by multiplying the filters by Gaussian weighting functions. The development of the ORTHO filters in the one-dimensional case is presented in Appendix A.2 for illustrating the basic idea of the method.

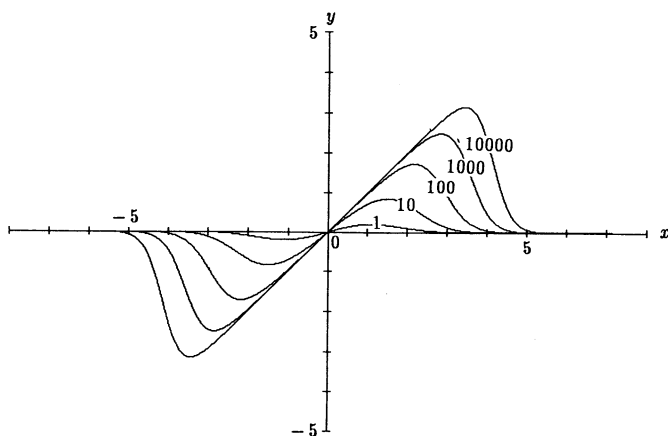


Fig. 1. The shapes of the Maximum Likelihood location filters for the Gaussian of $\sigma = 1$ of various signal-to-noise (S/N) ratios in one dimension. Each filter shows a gradual cut off at some distance from the center, and they are not the same shape as each other

2.2. Orthogonal functions in two dimensions

First, suppose the shape of the mean stellar image is a Gaussian whose variance is ρ^2 , i.e.

$$\bar{I}(r, \theta) = NG(r, \theta; \rho), \quad (1)$$

$$G(r, \theta; \rho) = \frac{1}{2\pi\rho^2} \exp\left(-\frac{r^2}{2\rho^2}\right), \quad (2)$$

where N is the total photon count from the star. Then, as is shown in Appendix A.2, it is convenient to develop a series of filters by using the orthogonal functions whose orthogonal relation is defined by Gaussian-weights. The orthonormal series $\{S_n^{(m)}(r, \theta)\}$ on the Gaussian weight are expressed with the associated Laguerre polynomials as follows:

$$S_n^{(m)}(r, \theta) = \sqrt{\frac{n!}{(n+m)!} \frac{(2+w^2)^{(m+1)/2}}{2^{m/2} w^{m+1}}} (-1)^n \times \\ \times L_n^{(m)}\left(\frac{2+w^2}{2w^2} r^2\right) r^m e^{im\theta} \exp\left(-\frac{r^2}{2w^2}\right), \quad (3)$$

where w is defined by the ratio of the characteristic width of the filter to ρ . Namely, if $w = 1$, the characteristic width of the filter is equal to the image size of the star. The (x, y) coordinates are regarded as a complex $z = x + iy = re^{i\theta}$.

Any component of the ORTHO filter is composed of a proper linear combination of the above orthogonal functions. Although the functions of only $m = 0$ and $m = 1$ are used in the simulation in Sect. 4, the overview of the functions for higher orders is also shown in Fig. 2.

2.3. Minimum variance filters

Let us assume that a filter to yield the estimator $\hat{\Theta}$ of a certain parameter Θ can be expressed with the above orthogonal functions as

$$F_{\Theta}^{(M)}(r, \theta; \rho) = \sum_{n=0}^M \sum_{m=-\infty}^{\infty} c_{mn} S_n^{(m)}\left(\frac{r}{\rho}, \theta\right), \quad (4)$$

where M is the order of filter. Since the functions are orthonormal, the variance $\sigma_{\hat{\Theta}}^2$ is immediately obtained (cf. Appendix A.2 for one-dimensional case):

$$\sigma_{\hat{\Theta}}^2 = \sum_{n=0}^M \sum_{m=-\infty}^{\infty} |c_{mn}|^2. \quad (5)$$

The coefficients $\{c_{mn}\}$ should satisfy the condition that the expected value of the estimator $E(\hat{\Theta})$ coincides with the true value Θ at $\tau \rightarrow 0$, where $\tau = x_c + iy_c$ is the position of the center of the Gaussian-distributed image in complex space. The coefficients $\{c_{mn}\}$ of the minimum variance filter are determined as a conditional minimum problem by means of Lagrange's method of indeterminate coefficients. The details of the procedure for the one-dimensional case are discussed in Appendix A.2. The scheme for the derivation of the minimum variance coefficients for the two-dimensional case is the same as that for the one-dimensional case.

The most interesting application is the derivation of the filter for photon counts and location estimations. The

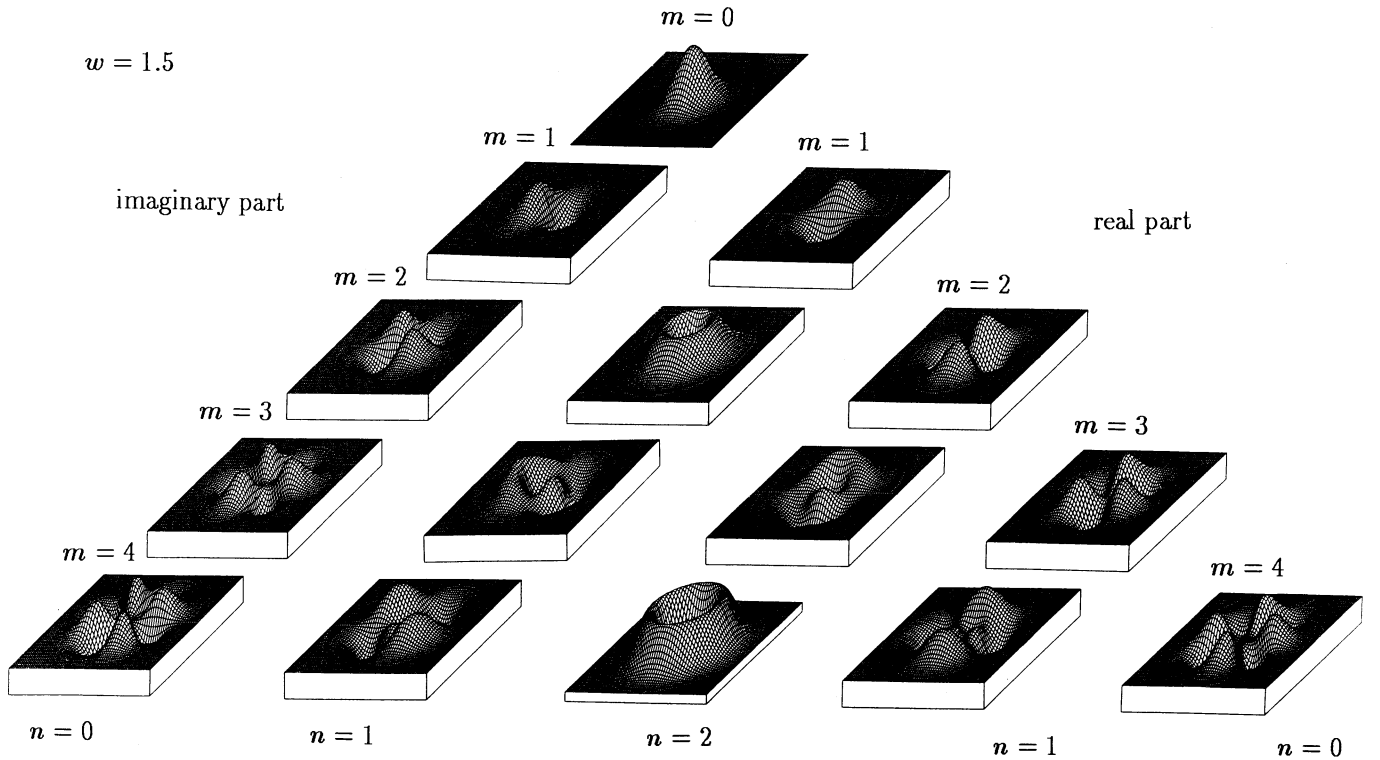


Fig. 2. The overview of the orthogonal functions $\{S_n^{(m)}(r, \theta)\}$ on the Gaussian weight in two dimensions. The first parts of the series from $m = 0$ to 4 are illustrated. The real part and the imaginary part correspond to the x and y components of the filter, respectively. Arbitrary filter estimators can be composed of a proper linear combination of these functions

exact expressions for the coefficients of the two filters are given in Appendix A.3. Specifically in the case of $M = 0$, the filters are given by the equations

$$\begin{aligned} F_N^{(0)}(r, \theta; \rho) &= \frac{1 + w^2}{w^2} \exp\left(-\frac{r^2}{2w^2\rho^2}\right) \quad (\text{photon counts}), \\ F_{N\tau}^{(0)}(r, \theta; \rho) &= \frac{(1 + w^2)^2}{w^4} r e^{i\theta} \exp\left(-\frac{r^2}{2w^2\rho^2}\right) \quad (\text{location}). \end{aligned} \quad (6)$$

The photon count filter for $M = 0$ is just a Gaussian as shown above.

2.4. Local orthogonality

The orthogonal functions obtained above compose the non-bias and minimum variance ORTHO filters, to be exact, in the sense of local at just one point of $\tau = 0$ and $\phi = \rho$ in the parameter space, where ϕ is the actual image size of a star. Therefore except this particular point, the functions are no longer orthogonal, and bias as well as degradation of the estimators are caused. Indeed, the photon count and location estimators of zeroth order become

$$\widehat{N} = N \exp\left(-\frac{|\tau|^2}{2(\phi^2 + w^2\rho^2)}\right) + O((\phi - \rho)^2) + O(\tau^4), \quad (7)$$

$$\widehat{N\tau} = N\tau \exp\left(-\frac{|\tau|^2}{2(\phi^2 + w^2\rho^2)}\right) + O((\phi - \rho)^2) + O(\tau^4). \quad (8)$$

The factor of $\exp\{-|\tau|^2/(2(\phi^2 + w^2\rho^2))\}$ comes from the non-orthogonality introduced by the decentered image. As this is a bias, the effect must be corrected. The situation is almost the same for other estimators (for $\widehat{\tau} = \widehat{N\tau}/\widehat{N}$, this factor cancels), therefore the center of the filter and the expected image size of the star should be as close to those of the actual one as possible. As the approximate position of the center of the stellar image is obtained with the location filter, the center of the filter can be placed close to the image center. The iterative scheme allows for a successive repositioning of the filter if the iteration converges and in this way the bias and the degradation of the estimator can be minimized.

3. Performance of the orthogonal filters

3.1. Accuracy of the estimator

As shown in Eq. (3), the orthogonal series in two dimensions $\{S_n^{(m)}(r, \theta)\}$ includes a parameter w , the ratio of characteristic width of the filter to the image size. By means of adjusting w to an appropriate value, the variance of the estimator can be minimized. When the Gaussian image in two dimensions is contaminated by a constant

background noise of density level B , i.e.

$$\bar{I}(x, y) = B + \frac{N}{2\pi\rho^2} \exp\left(-\frac{(x-x_c)^2 + (y-y_c)^2}{2\rho^2}\right), \quad (9)$$

then the variance of the photon count estimator and the location estimator for $M = 0$ is given as follows (Appendix A.3):

$$\sigma_N^2 = \frac{(1+w^2)^2}{w^2(2+w^2)} N \left(1 + \pi(2+w^2) \frac{B\rho^2}{N}\right), \quad (10)$$

$$\sigma_x^2 = \sigma_y^2 = \left(\frac{1+w^2}{w^4(2+w^2)^2}\right) \frac{\rho^2}{N} \left(1 + \frac{\pi(2+w^2)^2 B\rho^2}{2N}\right). \quad (11)$$

Differentiated by w , the optimal width w_{opt} which minimizes $\sigma_\tau^2 = \sigma_x^2 + \sigma_y^2$ is obtained approximately by

$$w_{opt} \approx \left(1 + \frac{4N}{\pi B\rho^2}\right)^{1/8}. \quad (12)$$

The variance of the photon count estimator σ_N^2 also takes a minimum approximately at this filter width. For ORTHO filters with $M > 0$, the optimal width decreases as M is increased, but the minimum value of the variance scarcely varies.

3.2. Comparison with other methods

In order to examine the effectiveness of the ORTHO estimators, we compare its performance with those of other methods. Table 1 shows the theoretical expectations of the variances of the location estimator of the Maximum Likelihood filter, the one- and two-dimensional ORTHO filters, and that of the Trimmed Moment filter. $L = 2 \times 3\rho$ is assumed for the Trimmed Moment filter and the marginal distribution of the one-dimensional ORTHO filter.

It is clear from Table 1 that the accuracies of the Trimmed Moment filter and the one-dimensional ORTHO filter are more degraded than that of the two-dimensional ORTHO filter at poor S/N ratios while there is no significant difference between them for good S/N ratios. The two-dimensional ORTHO filters are robust in the presence of noise.

The individual discussions follow.

3.2.1. Comparison with the ML estimator

The form of the ML filters for different signal-to-noise (S/N) ratios differ from each other (Fig. 1). On the other hand, in the application of ORTHO filters, it is sufficient to prepare only a few numerical filters for various S/N ratio images. In the ML estimation, the parameters to compose a filter are simultaneously estimated, hence the ML filter is variable. This fact requires a much longer time or procedure for the case of ML estimators than for the

Table 1. Comparison of the standard deviations of the location estimators by various methods for the case of $\rho = 1''$, $B = 1$ count/□'. S/N is defined by $N/B\rho^2$

S/N	Maximum Likelihood	2-D ORTHO ($M = 0$)			1-D ORTHO ($M = 0$)			Trimmed Mean
		$w = 1.0$	$w = 1.2$	$w = 1.5$	$w = 1.0$	$w = 1.2$	$w = 1.5$	
1	2 ⁿ .699	5 ⁿ .188	5 ⁿ .319	5 ⁿ .986	6 ⁿ .639	6 ⁿ .784	7 ⁿ .433	10 ⁿ .440
3	0.972	1.754	1.773	1.962	2.177	2.211	2.403	3.334
10	0.404	0.655	0.643	0.684	0.761	0.761	0.811	1.086
32	0.195	0.285	0.269	0.271	0.302	0.294	0.301	0.374
100	0.103	0.142	0.131	0.125	0.140	0.133	0.130	0.144
316	0.057	0.077	0.070	0.065	0.073	0.068	0.065	0.065
1000	0.032	0.042	0.038	0.035	0.040	0.037	0.035	0.033
3162	0.018	0.024	0.021	0.020	0.022	0.021	0.019	0.018
10000	0.010	0.013	0.012	0.011	0.012	0.011	0.011	0.010
31623	0.006	0.007	0.007	0.006	0.007	0.006	0.006	0.006
100000	0.003	0.004	0.004	0.003	0.004	0.004	0.003	0.003

ORTHO estimator. However an estimation of computing time depends on programming methods and algorithms, so we attempt to estimate it by the number of operations.

Let the filter size be $A \times A$ pixels. In the case of the ORTHO filter, we have only to prepare three filters, the photon count filter and the location (x - and y -coordinates) filters at the beginning of the procedure. In order to construct these filters, the exponential function is called $3A^2$ times. (This is for the most simple case: however, if we take the anti-bias integrator mentioned later in Sect. 4, it would be $3(2A+1)^2$ times.) After that, $3A^2$ calls of product and adding operations are required for the convolution of the image and the filters at each iteration of each object. The scheme is shown in Fig. 3. As the iteration converges in two or three cycles in most cases, the total number of operations is, typically, $6A^2 \sim 9A^2$ calls of product and adding operations.

On the other hand in the case of the ML method, since the filter shape depends on the estimated parameter N and B , the filter should be recalculated for every iteration and every object. Therefore the exponential function is called $3A^2$ times, and $3A^2$ calls of product and adding for convolution are required at every iteration of every object. As calling the exponential function takes a much longer time than product or adding, frequent calling of the exponential function makes it significantly slower.

In the two-dimensional image location problem, the variance of the ML location estimator for the Gaussian image is given as

$$\sigma_x^2 = \sigma_y^2 = \frac{\rho^2}{N} \left(1 + 2\pi \frac{B\rho^2}{N}\right). \quad (13)$$

(See Appendix A.3).

The performance of the ORTHO filters is slightly degraded compared to that of the ML estimation. The degree of the degradation depends upon the filter width w and the number of terms M , so we choose an appropriate combination of w and M for a particular problem. For the case of $M = 0$, the degradation is not larger than 10% at good S/N ratios.

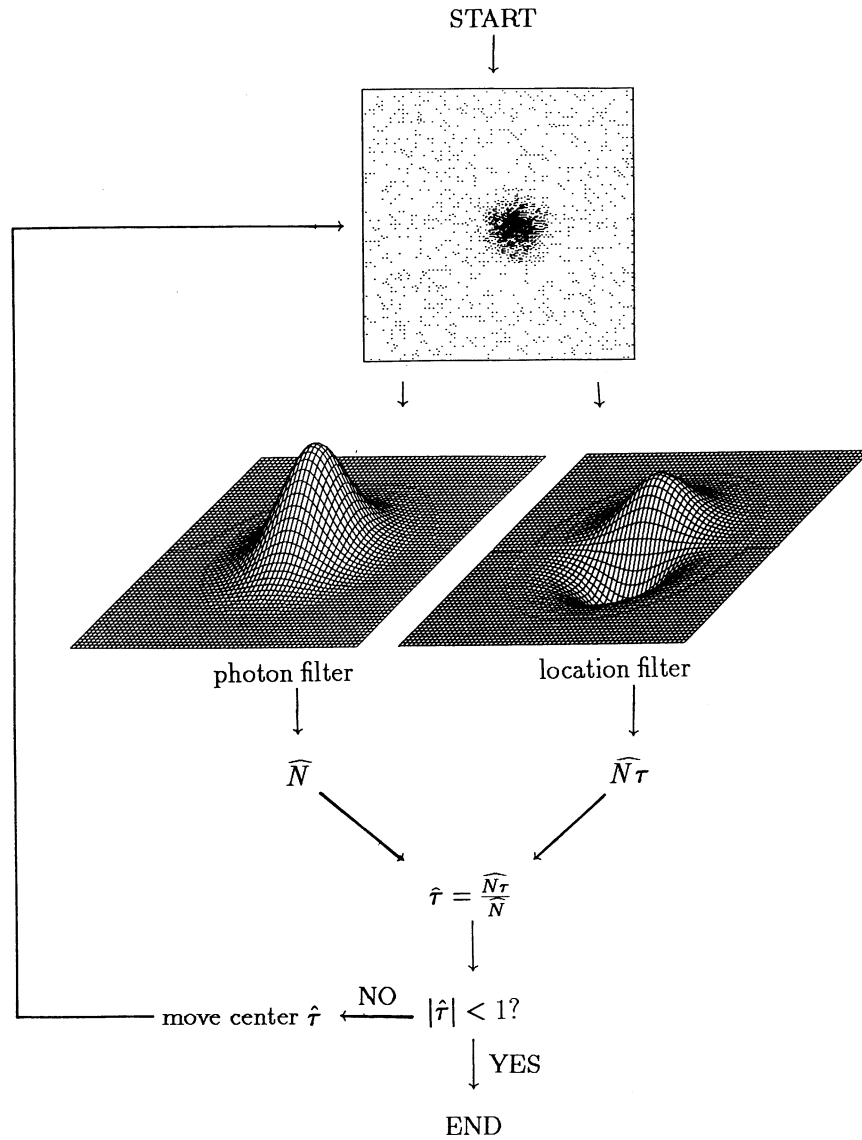


Fig. 3. The schematic diagram of the location estimation of a star. Two kinds of filters, the photon filter and the location filter, are superimposed on the stellar image, then the estimated values \hat{N} and $\hat{N}\hat{\tau}$ are obtained. The center of the stellar image $\hat{\tau}$ is estimated by $\hat{\tau} = \hat{N}\hat{\tau}/\hat{N}$. If $|\hat{\tau}|$ is not less than one pixel, move the filters by $\hat{\tau}$ and repeat above procedure until convergence. The procedure sometimes does not converge for poor images

3.2.2. Comparison with the Trimmed Moment estimator

One of the simplest filters would be the Trimmed Moment filters. The Trimmed Moment filter of the zeroth order is a rectangular photon count filter, and that of the first order is a Trimmed Mean filter.

The variances of the Trimmed Moment estimators of the zeroth and the first order are given, respectively, as follows:

$$\begin{aligned} \sigma_N^2 &= \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} (NG(r, \theta; \rho) + B) dy dx \\ &\approx N \left(1 + \frac{L^2 B \rho^2}{\rho^2 N} \right) \quad (L \gg \rho) \end{aligned} \tag{14}$$

$$\begin{aligned} \sigma_\tau^2 &= \frac{1}{N^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} x^2 (NG(r, \theta; \rho) + B) dy dx \\ &\approx \frac{\rho^2}{N} \left(1 + \frac{L^4 B \rho^2}{12 \rho^4 N} \right) \quad (L \gg \rho), \end{aligned} \tag{15}$$

where L is the box size. Apparently, the variances depend on the box size L . When L is taken too large, the accuracy of the estimation is degraded. This is caused by the contribution of background noise coming far from the image center. An adequate box size might be $L \approx 2 \times 3\rho$.

3.2.3. Comparison with the one-dimensional ORTHO estimator

One of the simplest approaches to two-dimensional astrometry is the application of a one-dimensional filter to a marginal distribution. The two-dimensional ORTHO filter is different from the one-dimensional ORTHO filter used with a marginal distribution. In the case of the one-dimensional ORTHO filter used with a marginal distribution, the photon count estimator and the location estimator are defined as follows:

$$\hat{N} = \int_{-\infty}^{\infty} f_N^{(M)}(x) \int_{-\infty}^{\infty} I(x, y) dy dx, \quad (16)$$

$$\hat{x} = \frac{1}{N} \int_{-\infty}^{\infty} f_{N\tau}^{(M)}(x) \int_{-\infty}^{\infty} I(x, y) dy dx, \quad (17)$$

$$\hat{y} = \frac{1}{N} \int_{-\infty}^{\infty} f_{N\tau}^{(M)}(y) \int_{-\infty}^{\infty} I(x, y) dx dy. \quad (18)$$

The details of the one-dimensional ORTHO filter are described in Appendix A.2. When $I(x, y) = NG(x, y; \rho) + B$ and $M = 0$, the variances of the estimators are given as follows:

$$\begin{aligned} \sigma_N^2 &= \int_{-\infty}^{\infty} |f_N^{(0)}(x)|^2 \int_{-L/2}^{L/2} (NG(x, y) + B) dy dx \\ &\approx \frac{1+w^2}{w\sqrt{2+w^2}} N \left(1 + \sqrt{\pi} \sqrt{2+w^2} \frac{B\rho L}{N} \right) \quad (L \gg \rho), \end{aligned} \quad (19)$$

$$\begin{aligned} \sigma_\tau^2 &= \frac{1}{N^2} \int_{-\infty}^{\infty} |f_{N\tau}^{(0)}(x)|^2 \int_{-L/2}^{L/2} (NG(x, y) + B) dy dx \\ &\approx \frac{(1+w^2)^3 \rho^2}{w^3(2+w^2)^{3/2} N} \left(1 + \frac{\sqrt{\pi}(2+w^2)^{3/2} B\rho L}{2N} \right) \quad (L \gg \rho), \end{aligned} \quad (20)$$

It is apparent that the variances depend on L , and the larger the filter size is, the worse the accuracy of the estimators are, because of the background noise contribution.

For the one-dimensional ORTHO filter with a marginal distribution, the number of operations can be somewhat less than that for a two-dimensional case. However it is difficult to apply it to the higher order estimators such as the eccentricity estimator, since the one-dimensional ORTHO filtering with a marginal distribution does not provide any information concerning the covariance between x and y of an asymmetric image.

3.3. Potential for extension to a non-Gaussian image

One of the original motives of the present filtering theory is the application of ORTHO filters to non-single Gaussian image distributions such as galaxies, planets, double stars and so on. In the application of the ML method, different model distributions are required according to the shape of

image under analysis. On the other hand, for the ORTHO filters, in principle no separate treatment is needed for any objects. Different types of image forms analyzed with the appropriate set of ORTHO filters can be classified according to the significance of the derived coefficients to their uncertainties.

Now we show a few examples. One of the first applications is the image size estimator for an axi-symmetric Gaussian. The filter for $M = 1$ (the lowest order) is composed by $S_0^{(0)}(r/\rho, \theta)$ and $S_1^{(0)}(r/\rho, \theta)$ as follows:

$$F_{N\phi^2}^{(1)}(r, \theta) = \frac{\rho^2(1+\rho^2)}{\sqrt{1+2\rho^2}} \left\{ (1-\rho^2) S_0^{(0)}\left(\frac{r}{\rho}, \theta\right) + (1+\rho^2) S_1^{(0)}\left(\frac{r}{\rho}, \theta\right) \right\}. \quad (21)$$

When the true image size of the Gaussian is $\phi^2 = \rho^2 + \Delta\rho^2$, the estimator becomes

$$\hat{\phi}^2 = \rho^2 + \Delta\rho^2 + O(\Delta\rho^4) \quad (\Delta\rho \rightarrow 0). \quad (22)$$

Hence the estimator is locally non-biased at $\phi^2 = \rho^2$. If we take $M = 2$ or larger, the residual term can be decreased to $O(\Delta\rho^6)$ or less.

Another application is the eccentricity estimator for an asymmetric Gaussian whose semi-axis lengths are a and b . Define

$$\Theta_n^{(m)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_n^{(m)}\left(\frac{r}{\rho}, \theta\right) I(x, y) dx dy, \quad (23)$$

where $I(x, y)$ is the distribution function of the image. In this case,

$$I(x, y) = \frac{N}{2\pi ab} \exp \left\{ -\frac{(x \cos P - y \sin P)^2}{2a^2} - \frac{(x \sin P + y \cos P)^2}{2b^2} \right\}, \quad (24)$$

where P is the orientation of the major axis. Then

$$\Theta_0^{(0)} = \frac{w\sqrt{2+w^2}\rho^2 ab}{\sqrt{(a^2+w^2\rho^2)(b^2+w^2\rho^2)}} N, \quad (25)$$

$$\Theta_0^{(2)} = -\frac{w^5(2+w^2)^{3/2}\rho^4 a^2}{(a^2+w^2\rho^2)^{3/2}(b^2+w^2\rho^2)^{3/2}} N \epsilon^2 e^{2iP}, \quad (26)$$

where $\epsilon \equiv \sqrt{a^2 - b^2}/a$ is the eccentricity. Hence the eccentricity and orientation of the ellipse can be obtained.

The number of operations required is almost the same as discussed in Sect. 3.2.

Further detail will be discussed in a future paper.

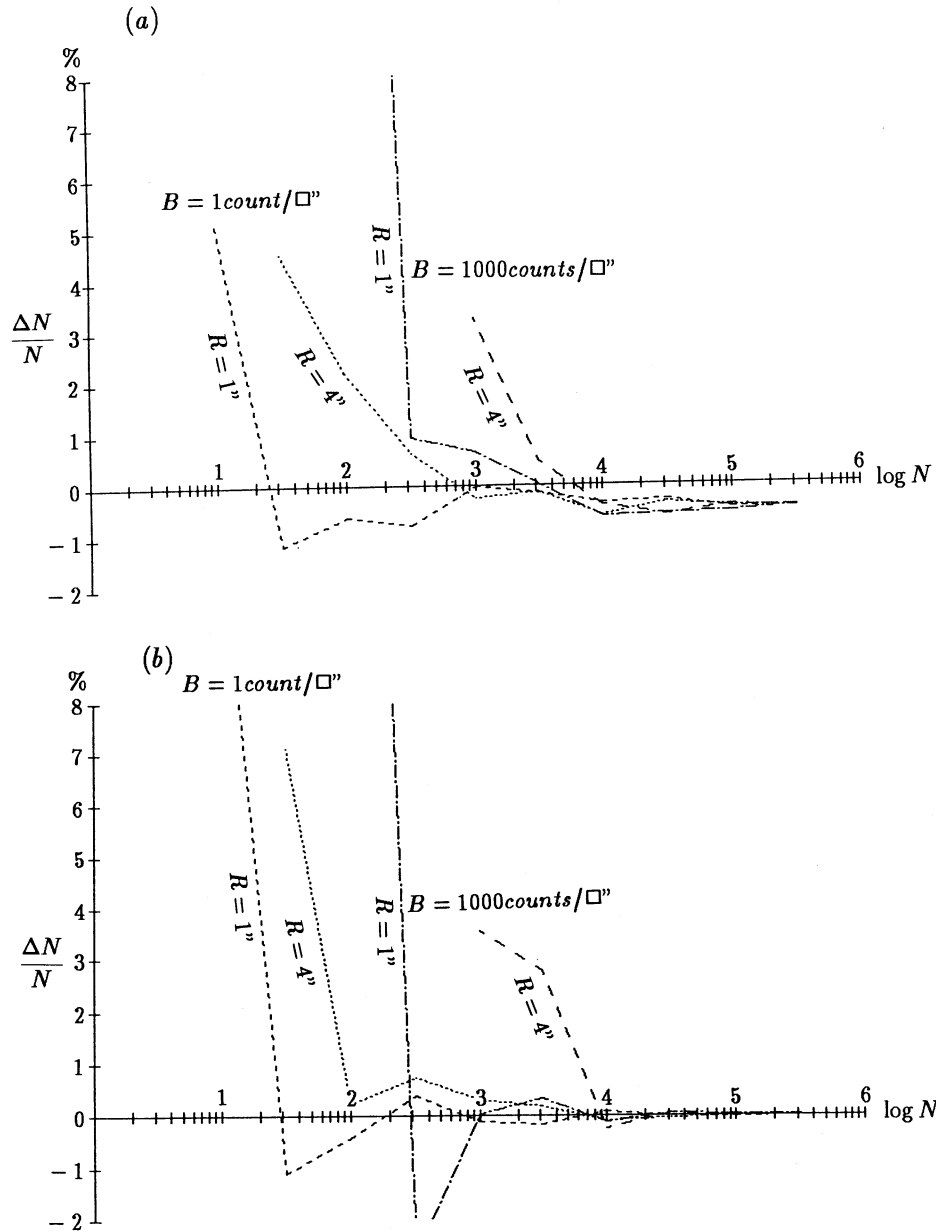


Fig. 4. a) and b). The simulated biases of the total photon count estimator by Simpson's integrator (Fig. 4a) and the anti-bias integrator (Fig. 4b). Four cases are simulated. Two kinds of biases, a negative constant bias at good S/N ratio and a positive bias at poor S/N ratio, are seen in all the cases in Fig. 4a. The cause of the positive bias is unknown but the negative bias is caused by the digitization effect as is discussed in the text. The negative bias is cancelled out by use of the anti-bias integrator (Fig. 4b)

4. Simulation

4.1. Procedure

In order to verify the theory described in the previous sections, numerical simulations were performed according to a scheme similar to that of Stone (1989). The shape of the simulated stellar image is a two-dimensional circular Gaussian contaminated by a flat sky-background, both with Poissonian noise. The size of the field of view is 64×64 pixels in all simulations. Filters with the same size

as the field of view were used throughout the reductions. The image size R of the simulated stars extends to 8 pixels in FWHM, which is related to the standard deviation ρ of the Gaussian distribution by

$$\rho = \frac{R}{2\sqrt{\ln 4}} \approx 0.425R. \quad (27)$$

The parameter w is fixed at 1.5. This choice is approximately optimal for any S/N ratio (cf. Eq. (A19) and Table 1). The parameters to be estimated are: 1) the

total number of photons in the image N , 2) the position of the image center $\tau = x_c + iy_c$. The image size R and the sky-background level B are supposed to be known, and the constant sky-background can be completely subtracted by preprocessing.

We made the simulation calculations for the cases of

$$\begin{cases} R &= 1 \text{ and } 4 \text{ arcsec,} \\ B &= 1 \text{ and } 1000 \text{ counts}/\square'', \\ \log N &= 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5 \text{ and } 5.5 \end{cases}$$

Since we know the behavior of the theoretical expectations, it is sufficient to find the accuracy of the estimators just for these cases, not for all of those in Stone's paper. The simulations were run 100 times for each case, compared to 25 times in Stone's paper.

The iterative scheme of the estimation process is illustrated in Fig. 3. The integration within a pixel for the filter values is performed with Simpson's integrator in two dimension (Fig. 5d). The estimator, \hat{N} and $\hat{N}\tau$, are evaluated by summing the product of the filter values and simulated photon counts over the whole pixels:

$$\hat{N} = \sum_{x=1}^{64} \sum_{y=1}^{64} F_N^{(0)}(x - x_c, y - y_c) I(x, y), \quad (28)$$

$$\hat{N}\tau = \sum_{x=1}^{64} \sum_{y=1}^{64} F_{N\tau}^{(0)}(x - x_c, y - y_c) I(x, y). \quad (29)$$

The image center τ is evaluated by

$$\hat{\tau} = \frac{\hat{N}\tau}{\hat{N}}. \quad (30)$$

When a new position for the image center is obtained, the center of the filter is moved to the nearest position, and \hat{N} and $\hat{N}\tau$ are evaluated again. If $|\hat{\tau}| > 1$ pixel then the procedure should be repeated until $|\hat{\tau}| < 1$ pixel. When the iteration did not converge in some cases, then the procedure was interrupted as the S/N ratio was too low.

4.2. Results

4.2.1. Photon count estimator

The total number of the photons from a star image is estimated by the photon count filter of zeroth order. The factor of $\exp\{-|\tau|^2/(2(1+w^2)\rho^2)\}$ needs to be corrected for decentered images.

The results of the simulation suggest two types of bias. One is the positive bias for poor S/N ratio and the other is the negative bias proportional to N for good S/N ratio. The origin of the positive bias is not clear at present but it may be statistical one. The value of the negative bias is about -0.4% of the total photon count N . The negative bias is thought to result from the digitization into pixels. The digitization bias is, essentially, the mathematical discrepancy between the integration of the product of two

smooth functions and the product of the two integrations of smooth functions.

$$\begin{aligned} & \frac{1}{a^4} \left\{ \iint_{\text{pixel}} f(x, y) dx dy \right\} \left\{ \iint_{\text{pixel}} g(x, y) dx dy \right\} \\ &= \frac{1}{a^2} \left\{ \iint_{\text{pixel}} f(x, y) g(x, y) dx dy + \Delta(x_0, y_0) \right\}, \end{aligned} \quad (31)$$

where $f(x, y)$ and $g(x, y)$ are arbitrary smooth functions, (x_0, y_0) is the center of the pixel, a is the pixel size, and $\Delta(x_0, y_0)$ is the digitization term:

$$\begin{aligned} \Delta(x_0, y_0) = & -\frac{a^2}{12} \left(\frac{\partial f(x_0, y_0)}{\partial x} \frac{\partial g(x_0, y_0)}{\partial x} \right. \\ & \left. + \frac{\partial f(x_0, y_0)}{\partial y} \frac{\partial g(x_0, y_0)}{\partial y} \right) \\ & + O(a^4). \end{aligned} \quad (32)$$

The total effect Δ of the digitization term for the Gaussian form of function is given by

$$\begin{aligned} \Delta &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta(x, y) dx dy \\ &= -\frac{w^3 \sqrt{2+w^2}}{6(1+w^2)^2} a^2. \end{aligned} \quad (33)$$

Substituting $w = 1.5$, $a = \sqrt{\ln 4}/4w$, we obtain $\Delta = -4.23 \cdot 10^{-3}$. This is quite consistent with the value found above.

The digitization bias cannot be removed regardless of what integrator we choose to apply within a pixel, since the bias includes the spatial derivatives of the integrated functions (Eq. (32)). However, when the anti-bias integrator of third order is applied to 2×2 pixels (Fig. 5), the digitization effect cancels to the third order (Appendix A.4). It was verified that this integrator eliminates the digitization effect (Fig. 4b).

As for the variance of the photon count estimator, it is shown that the results of the simulations are satisfactory and coincident with the theoretical expectations given in the previous section in all cases.

4.2.2. Location estimator

The position of the image center is estimated with two filters, x - and y -coordinate filters. The two filters are considered to be the real and imaginary parts of one complex filter of $m = 1$.

No significant systematic bias is observed in the simulated location, and the fluctuation of the deviation from the true position is almost coincident with the expected value of $10\% = 1/\sqrt{100}$. The digitization effect causes no bias in the case of the location estimator. The coincidence between the expectation of the theory and the result of the simulations is satisfactory in all cases examined.

The ORTHO location estimator is about ten percent poorer than that of the ML estimator at good S/N ratio,

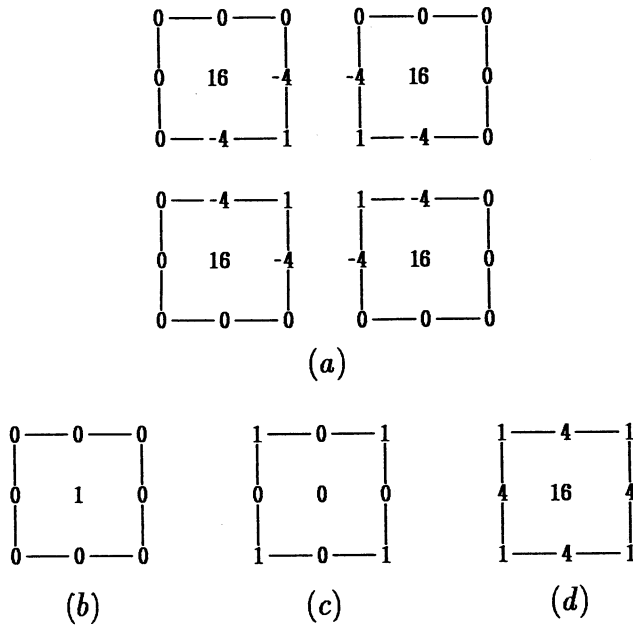


Fig. 5. a) the anti-bias integrator of the third order, b) the rectangular integrator, c) trapezoidal integrator, and d) Simpson's integrator for the filter in two dimensions, respectively. The integrators in one pixel (b)-(d) cause the digitization effect for the total photon count estimator while the anti-bias integrator cancels out this effect

but much poorer at low S/N ratios. This inferiority can be improved by decreasing w to, for example, 1.2 or 1.0. When w is decreased, the performance of the ORTHO estimator is improved at poor S/N ratios, but in contrast it is degraded at good S/N ratios. In order to resolve this dilemma, it is best to add another filter at good S/N ratios and develop an improved estimator for $M = 1$.

It is clear from Eqs. (11), (13), (15) and (20) that the variance of the location estimator increases in proportion to N^{-2} when B/N is large, whereas it behaves in proportion to N^{-1} for the case of negligible or zero background.

5. Conclusions

A new filtering technique for astrometry developed on the basis of orthogonal functions is presented, and a set of photon count and location filters are formulated for analyzing two-dimensional CCD images of stars. The filters are called ORTHO filters.

The theoretical accuracy of the filter estimators is presented and is confirmed by numerical simulations. It is found that location estimation with ORTHO filters is less accurate when compared with the ML method by about ten percent. However, the speed in computing and ease of handling make our new technique valuable in practical applications. Furthermore, it is more robust against noise than other non-ML filters.

A slight systematic bias is found for the photon count ORTHO filter which is caused by the digitization into pixels. Correction for non-centered image effect is also necessary to avoid a systematic bias with the photon count estimator. The magnitude of the non-centered image effect is confirmed to be comparable to that of the digitization effect by the simulation study.

It is possible to extend the present work to images which are not represented by a single Gaussian such as double stars, diffuse sources and so on. A set of new filters corresponding to these images will enable us to do automatic detection and classification of single stars, double stars, galaxies or moving objects. It is also important to develop a practical technique for the automatic distinction between a true image and a spurious one. Such distinction will improve the limiting magnitude.

Appendix

A.1 Basic concepts of filter estimators

Let $z = x + iy$ be the position of a pixel on a CCD array and $I(z) = I(x, y)$ be photon counts observed for a part of the sky containing one star. The expected value $\bar{I}(z; \tau)$ of $I(z)$ is modeled by using a function with the location of the star $\tau = x_c + iy_c$ being an unknown complex parameter to be estimated. A function W of the model distribution $\bar{I}(z; \tau)$ is called a *filter estimator* when the condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\bar{I}(z; \tau)) I(z) dx dy = 0 \quad (\text{A1})$$

yields an estimation $\hat{\tau}$ of the location parameter τ .

The arithmetic mean, or the first order moment, filter

$$W = z - \tau \quad (\text{A2})$$

is the simplest example of a filter estimator. For a star image modeled with a Gaussian distribution without any background contribution

$$\bar{I}(z; \tau) = \frac{N}{2\pi\rho^2} \exp\left(-\frac{|z - \tau|^2}{2\rho^2}\right), \quad (\text{A3})$$

we obtain

$$z - \tau = \frac{\rho^2}{\bar{I}(z; \tau)} \left(\frac{\partial \bar{I}(z; \tau)}{\partial x_c} + i \frac{\partial \bar{I}(z; \tau)}{\partial y_c} \right). \quad (\text{A4})$$

Then Eq. (A1) together with Eq. (A2) is equivalent to the equation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\bar{I}(z; \tau)} \left(\frac{\partial \bar{I}(z; \tau)}{\partial x_c} + i \frac{\partial \bar{I}(z; \tau)}{\partial y_c} \right) I(z) dx dy = 0. \quad (\text{A5})$$

This is no other than the equation that the ML location estimator satisfies (Lindgren 1978; Yoshizawa et al. 1985). Thus the arithmetic mean is equivalent to the ML estimator.

A.2 The orthogonal filters in one dimension

It would be convenient to describe the one-dimensional estimator for help in understanding the two-dimensional case.

First, suppose the distribution of the actual image of the star is $I(x)$. Second, suppose a parameter Θ of the image distribution is estimated by an estimator $\hat{\Theta}$ as

$$\hat{\Theta} = \int_{-\infty}^{\infty} f_{\Theta}(x)I(x)dx, \quad (\text{A6})$$

where $f_{\Theta}(x)$ is a filter for Θ .

Then the variance σ_{Θ}^2 of a certain estimator $\hat{\Theta}$ is defined as

$$\begin{aligned} \sigma_{\Theta}^2 &= E \left[\left(\hat{\Theta} - E[\hat{\Theta}] \right)^2 \right] \\ &= E \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (I(x) - \bar{I}(x))(I(x') - \bar{I}(x')) \right. \\ &\quad \times \left. f_{\Theta}(x)f_{\Theta}(x')dx dx' \right], \end{aligned} \quad (\text{A7})$$

where $\bar{I}(x)$ is the mean of $I(x)$. Since the photon noise process is Poissonian, the variance of the distribution is equal to its mean, i.e.

$$E \left[(I(x) - \bar{I}(x))(I(x') - \bar{I}(x')) \right] = \bar{I}(x)\delta(x - x'). \quad (\text{A8})$$

Therefore

$$\sigma_{\Theta}^2 = \int_{-\infty}^{\infty} |f_{\Theta}(x)|^2 \bar{I}(x)dx. \quad (\text{A9})$$

When the distribution is Gaussian, say, $\bar{I}(x) = Ng(x; \rho)$, where

$$g(x; \rho) = \frac{1}{\sqrt{2\pi\rho}} \exp\left(-\frac{x^2}{2\rho^2}\right), \quad (\text{A10})$$

then σ_{Θ}^2 is reduced to

$$\sigma_{\Theta}^2 = \int_{-\infty}^{\infty} |f_{\Theta}(x)|^2 Ng(x; \rho)dx. \quad (\text{A11})$$

In other words, the variance is the Gaussian-weighted norm. Hence when we expand a filter function $f_{\Theta}(x)$ by a series of orthogonal functions, it is convenient to use the orthogonal functions defined not in terms of the L^2 -norm but the Gaussian-weighted norm.

As is well known, the orthogonal series $\{s_n(x)\}$ on the Gaussian are expressed with the Hermite polynomials as follows:

$$s_n(x) = \sqrt{\frac{\sqrt{2+w^2}}{n!w}} H_n\left(\frac{\sqrt{2+w^2}}{w}x\right) \exp\left(-\frac{x^2}{2w^2}\right), \quad (\text{A12})$$

where w is the ratio of the characteristic width of the filter relative to the image size, and whose orthogonal relation is

$$\int_{-\infty}^{\infty} s_n(x)\overline{s_m(x)}g(x)dx = \delta_{nm}. \quad (\text{A13})$$

Any filters can be composed of the proper linear combinations of above functions. Hence suppose a filter $f_{\Theta}^{(M)}(x; \rho)$ to evaluate a certain model parameter Θ is

$$f_{\Theta}^{(M)}(x; \rho) = \sum_{n=0}^M c_n s_n\left(\frac{x}{\rho}\right). \quad (\text{A14})$$

Then the expected value of the estimator $\hat{\Theta}$ is

$$E(\hat{\Theta}) = \int_{-\infty}^{\infty} f_{\Theta}^{(M)}(x; \rho)Ng(x - \tau; \rho)dx, \quad (\text{A15})$$

and the coefficients $\{c_n\}$ must be determined so that they will satisfy the condition

$$E(\hat{\Theta}) = \Theta \quad (\tau \rightarrow 0). \quad (\text{A16})$$

This is reduced to

$$\begin{aligned} E(\hat{\Theta}) &= \frac{w}{\sqrt{1+w^2}} N \sum_{n=0}^{[M/2]} \left\{ \sqrt{\frac{(2n-1)!!}{(2n)!!}} \frac{c_{2n}}{(1+w^2)^n} \right. \\ &\quad \left. + \sqrt{\frac{(2n+1)!!}{(2n)!!}} \frac{w\sqrt{2+w^2}}{(1+w^2)^{n+1}} \frac{\tau}{\rho} c_{2n+1} \right\} + O(\tau^2). \end{aligned} \quad (\text{A17})$$

On the other hand, according to Eq. (A11), the variance of the estimator is

$$\begin{aligned} \sigma_{\Theta}^2 &= \sum_{n=0}^M \sum_{n'=0}^M \int_{-\infty}^{\infty} N c_n \overline{c_{n'}} s_n\left(\frac{x}{\rho}\right) \overline{s_{n'}\left(\frac{x}{\rho}\right)} g(x; \rho) dx \\ &= N \sum_{n=0}^M \sum_{n'=0}^M c_n \overline{c_{n'}} \delta_{nn'} \\ &= N \sum_{n=0}^M |c_n|^2. \end{aligned} \quad (\text{A18})$$

This is a natural result of the orthogonal series.

To minimize the variance σ_{Θ}^2 , $\{c_n\}$ are determined by means of Lagrange's method of indeterminate coefficients. Set

$$\Phi = \sum_{n=0}^M |c_n|^2 - \lambda(E(\hat{\Theta}) - \Theta), \quad (\text{A19})$$

where λ is the indeterminate coefficient, then the conditions

$$\begin{cases} \frac{\partial \Phi}{\partial c_n} = 2\overline{c_n} - \lambda \frac{\partial E(\hat{\Theta})}{\partial c_n} = 0 & (0 \leq n \leq M) \\ \frac{\partial \Phi}{\partial \lambda} = -E(\hat{\Theta}) + \Theta = 0 \end{cases} \quad (\text{A20})$$

should be satisfied for Eq. (A19) to be minimized.

Let's consider two special filters, the photon count filter $f_N^{(M)}(x; \rho)$ and the location filter $f_{N\tau}^{(M)}(x; \rho)$. Rewriting the expansion coefficient $\{c_n\}$ as $\{a_n\}$ and $\{b_n\}$ for $f_N^{(M)}(x; \rho)$ and $f_{N\tau}^{(M)}(x; \rho)$, respectively, we can derive the following equations:

$$f_N^{(M)}(x; \rho) = \sum_{n=0}^M a_n s_{2n}\left(\frac{x}{\rho}\right), \quad (\text{A21})$$

$$a_n = \frac{\left\{ \sqrt{\frac{(2n-1)!!}{(2n)!!}} \frac{1}{\sqrt{w}(2+w^2)^{1/4}(1+w^2)^{n-1/2}} \right\}}{\left\{ \sum_{k=0}^M \frac{(2k-1)!!}{(2k)!!} \frac{1}{(1+w^2)^{2k}} \right\}}, \quad (\text{A22})$$

and

$$f_{N\tau}^{(M)}(x; \rho) = \sum_{n=0}^M b_n s_{2n+1} \left(\frac{x}{\rho} \right), \quad (\text{A23})$$

$$b_n = \frac{\left\{ \sqrt{\frac{(2n+1)!!}{(2n)!!}} \frac{\rho}{w^{3/2}(2+w^2)^{3/4}(1+w^2)^{n-3/2}} \right\}}{\left\{ \sum_{k=0}^M \frac{(2k+1)!!}{(2k)!!} \frac{1}{(1+w^2)^{2k}} \right\}}, \quad (\text{A24})$$

respectively.

In the case of $M = 0$, the filters are reduced to

$$f_N^{(0)}(x; \rho) = \frac{\sqrt{1+w^2}}{w} \exp\left(-\frac{x^2}{2w^2\rho^2}\right), \quad (\text{A25})$$

$$f_{N\tau}^{(0)}(x; \rho) = \frac{(1+w^2)^{3/2}}{w^3} x \exp\left(-\frac{x^2}{2w^2\rho^2}\right), \quad (\text{A26})$$

respectively.

It can be shown that each filter converges to

$$\lim_{M \rightarrow \infty} f_N^{(M)}(x; \rho) = 1, \quad (\text{A27})$$

$$\lim_{M \rightarrow \infty} f_{N\tau}^{(M)}(x; \rho) = x, \quad (\text{A28})$$

asymptotically, which are the definitions of momenta of zeroth and first order, respectively.

Without any background contribution, the variance of a given estimator is defined by Eq. (A11). When the image is contaminated with a background contribution of density level B , the variance of the estimator changes from Eq. (A9) to

$$\sigma_{\Theta}^2 = \int_{-\infty}^{\infty} |f_{\Theta}(x)|^2 (Ng(x; \rho) + B) dx. \quad (\text{A29})$$

As for the case of the location estimator of $M = 0$, this is reduced to

$$\sigma_{\tau}^2 = \frac{(1+w^2)^3}{w^3(2+w^2)^{3/2}} \frac{\rho^2}{N} \left(1 + \frac{\sqrt{\pi}}{2} (2+w^2)^{3/2} \frac{B\rho}{N} \right). \quad (\text{A30})$$

To minimize σ_{τ}^2 , the condition

$$\frac{\partial \sigma_{\tau}^2}{\partial w} = \frac{(1+w^2)^2 (-4N + \sqrt{\pi}(w^2-1)(2+w^2)^{5/2} B\rho)}{w^4(2+w^2)^{5/2}} = 0 \quad (\text{A31})$$

is required. This equation is approximately solved as

$$w_{\text{opt}} \approx \left(1 + \frac{4N}{\sqrt{\pi} B\rho} \right)^{1/7}. \quad (\text{A32})$$

Hence the optimal basic width of the filter w should be varied according to S/N ratio.

The situation is almost the same for the photon count estimator. For $M = 0$, the variance σ_N^2 is given as

$$\sigma_N^2 = \frac{1+w^2}{w\sqrt{2+w^2}} N \left(1 + \sqrt{\pi} \sqrt{2+w^2} \frac{B\rho}{N} \right). \quad (\text{A33})$$

This is approximately minimized with w_{opt} given in Eq. (A32).

On the other hand, the ML location filter in one dimension is given as

$$f_{\tau}(x) = \frac{xg(x; \rho)}{g(x; \rho) + B/N}. \quad (\text{A34})$$

This is shown in Fig. 1.

Hence the variance of the ML location estimator is given as

$$\begin{aligned} \sigma_{\tau}^2 &= \int_{-\infty}^{\infty} \frac{x^2 \{g(x; \rho)\}^2}{g(x; \rho) + B/N} dx \\ &= \frac{\rho^2}{N} \left(1 + \sqrt{2\pi} \frac{B\rho}{N} \right). \end{aligned} \quad (\text{A35})$$

The calculation of this integral is a bit complicated.

A.3 Details of the orthogonal filters in two dimensions

The orthogonal series in two dimensions $\{S_n^{(m)}(r, \theta)\}$ given in Eq. (3) satisfies the following orthogonal relation:

$$\int_0^{\infty} \int_0^{2\pi} S_n^{(m)}(r, \theta) \overline{S_{n'}^{(m')}}(r, \theta) G(r, \theta) r^{m+1} d\theta dr = \delta_{nn'} \delta_{mm'}. \quad (\text{A36})$$

Since $\{S_n^{(m)}(r, \theta)\}$ is expressed with the associated Laguerre polynomials, the following formulae will be helpful for practical calculations. The scale conversion formula:

$$L_n^{(m)}(\beta^2 x^2) = \sum_{k=0}^n C_{k+m} \beta^{2k} (\beta^2 - 1)^{n-k} L_k^{(m)}(x^2). \quad (\text{A37})$$

The modulus conversion formula:

$$L_n^{(\alpha)}(x^2) = \begin{cases} \sum_{k=0}^{\min(n, \beta-\alpha)} \beta^{-\alpha} C_k (-1)^k L_{n-k}^{(\beta)}(x^2) & (\beta \geq \alpha) \\ \sum_{k=0}^{\min(n, \alpha-\beta)} \alpha^{-\beta} C_k L_{n-k}^{(\beta)}(x^2) & (\beta < \alpha) \end{cases}. \quad (\text{A38})$$

Furthermore,

$$\int_0^{2\pi} e^{z \cos \theta} e^{im\theta} d\theta = 2\pi I_m(z), \quad (\text{A39})$$

where $I_m(z)$ is the modified Bessel function of the m -th order, and

$$\int_0^{\infty} x^n \exp\left(-\frac{x^2}{2z^2}\right) dx = 2^{(n-1)/2} \Gamma\left(\frac{n+1}{2}\right) z^{n+1}, \quad (\text{A40})$$

where $\Gamma(z)$ is the Gamma function.

As is described in the text, the photon count filter and the location filter are composed of $\{S_n^{(m)}(r, \theta)\}$. The exact expressions are as follows:

$$F_N^{(M)}(r, \theta; \rho) = \sum_{n=0}^M a_n S_n^{(0)}\left(\frac{r}{\rho}, \theta\right), \quad (\text{A41})$$

$$a_n = \frac{w\sqrt{2+w^2}(1+w^2)^{2M-n+1}}{(1+w^2)^{2M+2}-1} \quad (\text{A42})$$

for the photon count filter, and

$$F_{N\tau}^{(M)}(r, \theta; \rho) = \sum_{n=0}^M b_n S_n^{(1)}\left(\frac{r}{\rho}, \theta\right), \quad (\text{A43})$$

$$b_n = \left\{ \frac{\sqrt{2}(1+w^2)^{2-n}\rho}{w^2(2+w^2)} \right\} / \left\{ \sum_{k=0}^M \frac{k+1}{(1+w^2)^{2k}} \right\} \quad (\text{A44})$$

for the location filter.

Similarly to the one-dimensional case, it can also be shown that

$$\lim_{M \rightarrow \infty} F_N^{(M)}(r, \theta; \rho) = 1, \quad (\text{A45})$$

$$\lim_{M \rightarrow \infty} F_{N\tau}^{(M)}(r, \theta; \rho) = re^{i\theta} = z \quad (\text{A46})$$

respectively.

The variances of the photon count estimator and the location estimator of $M = 0$ in the case of the Gaussian image smeared by constant background noise of density level B are derived similarly to the one-dimensional case,

$$\begin{aligned} \sigma_N^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_N^{(0)}(r, \theta; \rho)|^2 \left(NG(r, \theta; \rho) + B \right) dx dy \\ &= \frac{(1+w^2)^2}{w^2(2+w^2)} \left(1 + \pi(2+w^2) \frac{B\rho^2}{N} \right), \end{aligned}$$

$$\begin{aligned} \sigma_x^2 = \sigma_y^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2N^2} |F_{N\tau}^{(0)}(r, \theta; \rho)|^2 \times \\ &\quad \times (NG(r, \theta; \rho) + B) dx dy \\ &= \frac{(1+w^2)^4 \rho^2}{w^4(2+w^2)^2 N} \left(1 + \frac{\pi(2+w^2)^2 B\rho^2}{2N} \right). \end{aligned}$$

On the other hand, the variance of the ML location estimator is as follows:

$$\begin{aligned} \sigma_x^2 = \sigma_y^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x^2 |G(r, \theta; \rho)|^2}{(G(r, \theta; \rho) + B/N)} dx dy \\ &= \frac{\rho^2}{N} \left(1 + 2\pi \frac{B\rho^2}{N} \right). \end{aligned}$$

The calculation of this integral is also a bit complicated.

A.4 The theory of the anti-bias integrator

As is mentioned in Sect. 4.2.1, the digitization effect causes a negative bias for the photon count estimator. Now consider the one-dimensional case for simplicity.

Let a be the pixel size, then the observable of the total photon count within the pixel at x_0 is given as

$$N(x_0) = \frac{1}{a} \int_{x_0-a/2}^{x_0+a/2} g(x) dx, \quad (\text{A47})$$

where $g(x)$ is the photon count distribution (not a Gaussian). If $g(x)$ can be expanded as

$$\begin{aligned} g(x) &= g(x_0) + (x-x_0)g'(x_0) + \frac{(x-x_0)^2}{2!}g''(x_0) \\ &\quad + \frac{(x-x_0)^3}{3!}g'''(x_0) + \dots, \end{aligned} \quad (\text{A48})$$

then

$$N(x_0) = g(x_0) + \frac{a^2}{24}g''(x_0) + O(a^4). \quad (\text{A49})$$

Similarly, the filter value $F(x_0)$ on the pixel whose distribution is $f(x)$ becomes

$$F(x_0) = f(x_0) + \frac{a^2}{24}f''(x_0) + O(a^4). \quad (\text{A50})$$

And the correct value of an estimator Θ on the pixel is

$$\begin{aligned} \Theta(x_0) &= \frac{1}{a} \int_{x_0-a/2}^{x_0+a/2} f(x)g(x) dx \\ &= f(x_0)g(x_0) + \frac{a^2}{24} \left\{ f(x_0)g''(x_0) + 2f'(x_0)g'(x_0) \right. \\ &\quad \left. + f''(x_0)g(x_0) \right\} + O(a^4). \end{aligned} \quad (\text{A51})$$

On the other hand, the product of Eq. (A49) and Eq. (A50) becomes

$$\begin{aligned} \widehat{\Theta}(x_0) &= F(x_0)N(x_0) \\ &= f(x_0)g(x_0) + \frac{a^2}{24} \left\{ f(x_0)g''(x_0) \right. \\ &\quad \left. + f''(x_0)g(x_0) \right\} + O(a^4). \end{aligned} \quad (\text{A52})$$

This is not equal to Eq. (A51), and their difference

$$\Delta(x_0) = -\frac{a^2}{12}f'(x_0)g'(x_0) + O(a^4) \quad (\text{A53})$$

is the digitization term.

Since $N(x_0)$ is given *a priori* as an observable, we have to modify the integrator for $F(x_0)$ in order to cancel out the digitization term. But the spatial differential $g'(x_0)$ included in $\Delta(x_0)$ suggests that any integrator within a

pixel such as Simpson's integrator never cancels out the term.

So we set the integrator on two pixels which are located at $x = x_0 \pm a/2$ as follows:

$$\begin{cases} \tilde{F}(x_0 - a/2) = w_{00}f(x_0 - a) + w_{01}f(x_0 - a/2) + w_{02}f(x_0) \\ \tilde{F}(x_0 + a/2) = w_{10}f(x_0) + w_{11}f(x_0 + a/2) + w_{12}f(x_0 + a) \end{cases} \quad (\text{A54})$$

where $\{w_{ij}\}$ is the weight set. Then $\{w_{ij}\}$ is determined so that the following equation can be satisfied:

$$\begin{aligned} & \tilde{F}(x_0 - \frac{a}{2})N(x_0 - \frac{a}{2}) + \tilde{F}(x_0 + \frac{a}{2})N(x_0 + \frac{a}{2}) \\ &= \Theta(x_0 - \frac{a}{2}) + \Theta(x_0 + \frac{a}{2}) \end{aligned} \quad (\text{A55})$$

By means of a Taylor expansion around $x = x_0$ and a comparison of the coefficients, the following equations are obtained up to the third order:

$$\begin{cases} w_{00} + w_{01} + w_{02} + w_{10} + w_{11} + w_{12} = 2 \\ -w_{00} - w_{01} - w_{02} + w_{10} + w_{11} + w_{12} = 0 \\ -2w_{00} - w_{01} + w_{11} + 2w_{12} = 0 \\ 2w_{00} + w_{01} + w_{11} + w_{12} = 8/3 \\ 4w_{00} + w_{01} + w_{11} + 4w_{12} = 8/3 \\ -4w_{00} - w_{01} + w_{11} + 4w_{12} = 0 \\ -8w_{00} - w_{01} + w_{11} + 8w_{12} = 0 \end{cases} \quad (\text{A56})$$

$$\begin{pmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} \\ w_{10} & w_{11} & w_{12} & w_{13} & w_{14} \\ w_{20} & w_{21} & w_{22} & w_{23} & w_{24} \\ w_{30} & w_{31} & w_{32} & w_{33} & w_{34} \end{pmatrix} = \frac{1}{675} \begin{pmatrix} -4659 & 14624 & -14808 & 6240 & -722 \\ 8596 & -24448 & 20928 & -2176 & -2225 \\ -2225 & -2176 & 20928 & -24448 & 8596 \\ -722 & 6240 & -14808 & 14624 & -4659 \end{pmatrix} \quad (\text{A59})$$

But it is rather complicated and impractical.

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From the first five equations, the following solution is obtained:

$$\begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{10} & w_{11} & w_{12} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -\lambda & 4 + 2\lambda & -1 - \lambda \\ -1 + \lambda & 4 - 2\lambda & \lambda \end{pmatrix}, \quad (\text{A57})$$

where λ is an arbitrary constant. This is the anti-bias integrator of the second order.

If we take $\lambda = 0$, then the last two equations of Eq. (A56) are also satisfied. Therefore the solution becomes the anti-bias integrator of the third order:

$$\begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{10} & w_{11} & w_{12} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 & 4 & -1 \\ -1 & 4 & 0 \end{pmatrix}. \quad (\text{A58})$$

In two dimensions, the anti-bias integrator of the third order is shown in Fig. 5a.

Similarly, the anti-bias integrator of the fifth order can be composed on four pixels: