

A NEW ANOMALY OF KEPLERIAN MOTION

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ABSTRACT

A new anomaly of Keplerian motion, *projective anomaly*, is derived. The anomaly is derived from a resolution of one of the singularities of Keplerian motion from the viewpoint of projective geometry. Using the projective anomaly, every kind of Keplerian orbit—circular, elliptical, parabolic, hyperbolic, and two-body collisional linear orbit—can be treated in a unified way as a quadratic curve in the projective space in two dimensions. The projective anomaly is further shown to be an instance of a generalized anomaly.

Key words: celestial mechanics, stellar dynamics

1. INTRODUCTION

It is well known that three anomalies, the mean anomaly, the eccentric anomaly, and the true anomaly, are used in a traditional solution of Keplerian motion. However, there are two kinds of singularity in the traditional solution: namely, its expression is singular in the cases of two-body collision, and a parametric expression of an orbit with a traditional anomaly and form of Kepler's equation is not continuous at $e = 1$. In order to resolve the singularities, a study from the viewpoint of projective geometry is required.

2. KEPLERIAN ORBIT AS A QUADRATIC CURVE IN PROJECTIVE SPACE

From the viewpoint of projective geometry, all kinds of Keplerian orbit—circular, elliptical, parabolic, hyperbolic, and linear—are the same as quadratic curves. Consider a Keplerian orbit in a plane, given by

$$x = \frac{q(1+e)\cos f}{1+e\cos f}, \quad y = \frac{q(1+e)\sin f}{1+e\cos f}, \quad (1)$$

where q is the perihelion distance, e is the eccentricity, and f is the true anomaly. Eliminating f , we obtain an ordinary quadratic form for the orbit,

$$(1+e)[(1-e)x + eq]^2 + (1-e)y^2 = (1+e)q^2. \quad (2)$$

Apparently, this expression is singular when $e = 1$. Replacing (x, y) by $x = x_1/x_0$ and $y = x_2/x_0$ using homogeneous coordinates (x_0, x_1, x_2) in the projective space \mathbf{P}^2 , we have

$$-(1+e)qx_0^2 + 2ex_0x_1 + \frac{1-e^2}{(1+e)q}x_1^2 + \frac{1}{(1+e)q}x_2^2 = 0. \quad (3)$$

This can also be written in a quadratic form as

$$(x_0, x_1, x_2) \begin{bmatrix} -(1+e)q & e & 0 \\ e & (1+e)p & 0 \\ 0 & 0 & [(1+e)q]^{-1} \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0, \quad (4)$$

where $p = 1/Q = (1-e)/[(1+e)q]$, i.e., p is the reciprocal of the aphelion distance Q . Hence the characteristic equation is

$$[(1+e)q\lambda - 1][\lambda^2 - (1+e)(p-q)\lambda - 1] = 0; \quad (5)$$

its eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}[-(1+e)(q-p) \pm \sqrt{(1+e)^2(p+q)^2 + 4e^2}], \quad (6)$$

$$\lambda_0 = 1/(1+e)q,$$

where the relation $\lambda_+\lambda_- = -1$ is apparently found.

Now we set

$$\cosh \phi \cos 2\psi = \frac{1}{2}(1+e)(q+p), \quad \cosh \phi \sin 2\psi = e, \quad (7)$$

$$\sinh \phi = \frac{1}{2}(1+e)(q-p).$$

Then the eigenvalues are $\lambda_+ = e^{-\phi}$ and $\lambda_- = -e^{\phi}$, and their eigenvectors are

$$\lambda_+ = (\sin \psi, \cos \psi, 0), \quad \lambda_- = (\cos \psi, -\sin \psi, 0), \quad (8)$$

$$\lambda_0 = (0, 0, 1),$$

where

$$\cos \psi = \frac{1}{2}[(1+e)(q+p) + \sqrt{(1+e)^2(q+p)^2 + 4e^2}]q, \quad (9)$$

$$\sin \psi = e.$$

Therefore, a parametric expression for the elliptical cone of a Keplerian orbit in \mathbf{P}^2 is

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = (\lambda_-, \lambda_+, \lambda_0) \begin{pmatrix} 1/\sqrt{-\lambda_-} \\ \cos \theta/\sqrt{\lambda_+} \\ \sin \theta/\sqrt{\lambda_0} \end{pmatrix} = \exp\left(-\frac{\phi}{2}\right) \cos \psi \begin{pmatrix} 1 + \exp \phi \tan \psi \cos \theta \\ -\tan \psi + \exp \phi \cos \theta \\ \sqrt{\exp 2\phi - \tan^2 \psi} \sin \theta \end{pmatrix}, \quad (10)$$

where θ is the *projective anomaly*. Therefore, substituting

$$\alpha = \exp \phi = \frac{(1+e)(q-p) + \sqrt{(1+e)^2(q+p)^2 + 4e^2}}{2},$$

$$\beta = \tan \psi = \frac{2e}{(1+e)(q+p) + \sqrt{(1+e)^2(q+p)^2 + 4e^2}}, \quad (11)$$

we obtain

$$(x_0 : x_1 : x_2) = (1 + \alpha\beta \cos \theta : -\beta + \alpha \cos \theta : \sqrt{\alpha^2 - \beta^2} \sin \theta), \tag{12}$$

where α is the semimajor axis of the ellipse in \mathbf{P}^2 and β is a coordinate of the center of the ellipse in \mathbf{P}^2 .

3. EXPRESSION OF KEPLERIAN ELEMENTS

Some Keplerian elements are expressed by the parameters α and β as follows: given

$$\begin{aligned} x &= \frac{x_1}{x_0} = \frac{-\beta + \alpha \cos \theta}{1 + \alpha\beta \cos \theta}, \\ y &= \frac{x_2}{x_0} = \frac{\sqrt{\alpha^2 - \beta^2} \sin \theta}{1 + \alpha\beta \cos \theta}, \\ r &= \frac{\sqrt{x_1^2 + x_2^2}}{x_0} = \frac{\alpha - \beta \cos \theta}{1 + \alpha\beta \cos \theta}, \end{aligned} \tag{13}$$

we have

$$a = \frac{\alpha(1 + \beta^2)}{1 - \alpha^2\beta^2}, \quad e = \frac{\beta(1 + \alpha^2)}{\alpha(1 + \beta^2)}, \tag{14}$$

$$1 - e = \frac{(\alpha - \beta)(1 - \alpha\beta)}{\alpha(1 + \beta^2)}, \tag{15}$$

$$1 + e = \frac{(\alpha + \beta)(1 + \alpha\beta)}{\alpha(1 + \beta^2)}, \tag{16}$$

$$q = \frac{\alpha - \beta}{1 + \alpha\beta}, \quad p = \frac{1}{Q} = \frac{1 - \alpha\beta}{\alpha + \beta}. \tag{17}$$

The form of the orbit is circular for $\beta = 0$, elliptical for $\alpha\beta < 1$, parabolic for $\alpha\beta = 1$, hyperbolic for $\alpha\beta > 1$, and linear for $\alpha = \beta$.

4. KEPLER'S EQUATION

Kepler's equation for the projective anomaly θ is obtained by using the equation of energy. Substituting the formulae from the previous section into the equation of energy,

$$\frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] - \frac{k^2}{r} = -\frac{k^2}{2a}, \tag{18}$$

we obtain

$$\frac{d\theta}{dt} \frac{\alpha - \beta \cos \theta}{(1 + \alpha\beta \cos \theta)^2} = \frac{r}{x_0} = \frac{k}{\sqrt{\alpha(1 + \beta^2)}}. \tag{19}$$

This is integrated as

$$\int \frac{\alpha - \beta \cos \theta}{(1 + \alpha\beta \cos \theta)^2} d\theta = \frac{k(t - T_0)}{\sqrt{\alpha(1 + \beta^2)}}. \tag{20}$$

If $\alpha\beta < 1$ (an elliptical orbit), the integral is reduced by the substitutions

$$M = \left[\frac{1 - \alpha^2\beta^2}{\alpha(1 + \beta^2)} \right]^{3/2} k(t - T_0), \quad \tan \frac{\theta}{2} = \sqrt{\frac{1 + \alpha\beta}{1 - \alpha\beta}} \tan \frac{u}{2} \tag{21}$$

into an ordinary form of Kepler's equation,

$$u - e \sin u = M, \tag{22}$$

where u is the eccentric anomaly.

If $\alpha\beta = 1$ (a parabolic orbit), the integral is reduced by the substitution

$$s = \tan \frac{1}{2}\theta \tag{23}$$

into a cubic equation,

$$s^3 + 3 \frac{\alpha^2 - 1}{\alpha^2 + 1} s = \frac{6k(t - T_0)}{\sqrt{\alpha(\alpha^2 + 1)^{3/2}}}. \tag{24}$$

If $\alpha\beta > 1$ (a hyperbolic orbit), the integral is reduced by the substitutions

$$M' = \left[\frac{\alpha^2\beta^2 - 1}{\alpha(1 + \beta^2)} \right]^{3/2} k(t - T_0), \quad \tan \frac{\theta}{2} = \sqrt{\frac{\alpha\beta + 1}{\alpha\beta - 1}} \operatorname{tanh} \frac{u'}{2} \tag{25}$$

into an ordinary form of Kepler's equation,

$$e \sinh u' - u' = M', \tag{26}$$

where u' is the eccentric anomaly.

5. GENERALIZED ANOMALY

The relation between the eccentric anomaly u and the projective anomaly θ of equation (21) suggests a generalized anomaly Θ , related by

$$\tan \frac{1}{2}\Theta = \lambda \tan \frac{1}{2}u, \tag{27}$$

for arbitrary constant λ . In fact, Keplerian motion can be expressed as

$$\begin{aligned} x &= \frac{[\lambda^2 - (1 + e)/(1 - e)] + [\lambda^2 + (1 + e)/(1 - e)] \cos \Theta}{(\lambda^2 + 1) + (\lambda^2 - 1) \cos \Theta} q, \\ y &= \frac{2\sqrt{(1 + e)/(1 - e)} \lambda \sin \Theta}{(\lambda^2 + 1) + (\lambda^2 - 1) \cos \Theta} q, \\ r &= \frac{[\lambda^2 + (1 + e)/(1 - e)] + [\lambda^2(1 + e)/(1 - e)] \cos \Theta}{(\lambda^2 + 1) + (\lambda^2 - 1) \cos \Theta} q. \end{aligned} \tag{28}$$

Thus, the generalized anomaly becomes the eccentric anomaly for $\lambda = 1$, the true anomaly for $\lambda = [(1 + e)/(1 - e)]^{1/2}$, and the projective anomaly for $\lambda = [(1 + \alpha\beta)/(1 - \alpha\beta)]^{1/2}$.

6. CONCLUSION

From the viewpoint of projective geometry, a new anomaly, *projective anomaly*, has been derived. The projective anomaly resolves a singularity in the expression of a linear orbit, which is one of the singularities in Keplerian motion. The eccentric anomaly, the true anomaly, and the projective anomaly have been shown to be special cases of a generalized anomaly.