

# Generalized Jacobi Integral of the General Restricted Three-Body Problem

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## Abstract

Generalized Jacobi integral of the general restricted three-body problem is discussed. The integral has an additional term to Jacobi integral of the circular orbit case. The term is vanished in the case of the circular orbit. So the term is named “eccentric kinetic integral”.

**Key words:** celestial mechanics

## 1. introduction

The three-body problem has not been solved in general for long time since Isaac Newton discovered the universal gravitation in 1665. The essential reason is a singularity of simultaneous three-body collision. Namely, the orbits of three bodies are not analytically connectable before and after the simultaneous three-body collision. This difficulty also lies in the restricted three-body problem. The only algebraic integral of the restricted three-body problem ever known is Jacobi integral in the case of circular orbit discovered in 1836. Also several special solutions such as Lagrange’s stationary solutions have been known.

In one-dimensional restricted three-body problem, number of free parameters is two. In this case, the dynamic system can be solved if one of two integrals is known. However, one-dimensional case seems to be difficult to treat analytically because the singularity of two-body collisions occur many times. But this difficulty can be resolved using the projective anomaly. Originally, the projective anomaly was discovered to resolve the singularity of two-body collision. Also, all kind of orbit, circular, elliptical, parabolic, hyperbolic, and even linear can be treated unified way by the projective anomaly. Therefore the general restricted three-body problem in three-dimension can be solved without the difficulty of the singularity of two-body collisions.

## 2. equations of motion in three dimension

Keplerian motion in three dimension is expressed with the projective anomaly  $\theta$  as follows (Sato 1998 ),

$$\begin{aligned}\kappa_x &= \frac{-\beta + \alpha \cos \theta}{1 + \alpha\beta \cos \theta} = \frac{\sqrt{(1+e)q} - \kappa_r}{e}, \\ \kappa_y &= \frac{\sqrt{\alpha^2 - \beta^2} \sin \theta}{1 + \alpha\beta \cos \theta},\end{aligned}\tag{1}$$

$$\kappa_z = 0,$$

$$\kappa_r = \frac{\alpha - \beta \cos \theta}{1 + \alpha\beta \cos \theta},$$

where  $(\kappa_x, \kappa_y)$  are two-body coordinates,  $\kappa_r$  is relative distance of the two bodies,  $\alpha$  and  $\beta$  are parameters,

$$\alpha = \frac{(1+e)(q-p) + \sqrt{(1+e)^2(q+p)^2 + 4e^2}}{2},\tag{2}$$

$$\beta = \frac{2e}{(1+e)(q+p) + \sqrt{(1+e)^2(q+p)^2 + 4e^2}},$$

where  $e$  is the eccentricity,  $q$  is the perihelion distance, and  $p = 1/Q$  is the reciprocal of the aphelion distance, respectively. Let the masses of the two perturbing bodies be  $m_1 = 1 - \mu$  and  $m_2 = \mu$ , respectively. Then the equation of motion of a test particle in barycentric system are

$$\begin{cases} \frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \\ \frac{d^2y}{dt^2} = -\frac{\partial V}{\partial y} \\ \frac{d^2z}{dt^2} = -\frac{\partial V}{\partial z} \end{cases},\tag{3}$$

where  $V$  is potential,

$$V = -\frac{k^2(1-\mu)}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + z^2}} - \frac{k^2\mu}{\sqrt{(x-x_2)^2 + (y-y_2)^2 + z^2}} = -\frac{k^2(1-\mu)}{\delta_1} - \frac{k^2\mu}{\delta_2},\tag{4}$$

where  $k = \sqrt{G(m_1 + m_2)}$  is Gauss's gravitational constant,  $(x_1, y_1, 0) = (-\mu\kappa_x, -\mu\kappa_y, 0)$ ,  $(x_2, y_2, 0) = ((1-\mu)\kappa_x, (1-\mu)\kappa_y, 0)$ , and  $(x, y, z)$  are the coordinates of the two perturbing bodies and the test particle, respectively. In order to fix the two perturbing bodies from  $(-\mu\kappa_x, -\mu\kappa_y, 0)$  and  $((1-\mu)\kappa_x, (1-\mu)\kappa_y, 0)$  to  $(-\mu, 0, 0)$  and  $(1-\mu, 0, 0)$ , convert from the non-rotating coordinates  $(x, y, z)$  to the rotating coordinates  $(X, Y, Z)$  by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \kappa_x & -\kappa_y & 0 \\ \kappa_y & \kappa_x & 0 \\ 0 & 0 & \kappa_r \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.\tag{5}$$

Then Eq.(3) is reduced to

$$\begin{cases} \kappa_x \frac{d^2 X}{dt^2} - \kappa_y \frac{d^2 Y}{dt^2} + 2 \frac{d\kappa_x}{dt} \frac{dX}{dt} - 2 \frac{d\kappa_y}{dt} \frac{dY}{dt} + \frac{d^2 \kappa_x}{dt^2} X - \frac{d^2 \kappa_y}{dt^2} Y = -\frac{\partial V}{\partial x}, \\ \kappa_y \frac{d^2 X}{dt^2} + \kappa_x \frac{d^2 Y}{dt^2} + 2 \frac{d\kappa_y}{dt} \frac{dX}{dt} + 2 \frac{d\kappa_x}{dt} \frac{dY}{dt} + \frac{d^2 \kappa_y}{dt^2} X + \frac{d^2 \kappa_x}{dt^2} Y = -\frac{\partial V}{\partial y}, \\ \kappa_r \frac{d^2 Z}{dt^2} + 2 \frac{d\kappa_r}{dt} \frac{dZ}{dt} + \frac{d^2 \kappa_r}{dt^2} Z = -\frac{\partial V}{\partial z}, \end{cases} \quad (6)$$

where

$$V = -\frac{k^2}{\kappa_r} \left( \frac{1-\mu}{\Delta_1} + \frac{\mu}{\Delta_2} \right), \quad (7)$$

$$\frac{\partial V}{\partial x} = \frac{k^2}{\kappa_r^3} \left( \frac{(1-\mu)(\kappa_x(X+\mu) - \kappa_y Y)}{\Delta_1^3} + \frac{\mu(\kappa_x(X-1+\mu) - \kappa_y Y)}{\Delta_2^3} \right), \quad (8)$$

$$\frac{\partial V}{\partial y} = \frac{k^2}{\kappa_r^3} \left( \frac{(1-\mu)(\kappa_y(X+\mu) + \kappa_x Y)}{\Delta_1^3} + \frac{\mu(\kappa_y(X-1+\mu) + \kappa_x Y)}{\Delta_2^3} \right), \quad (9)$$

$$\frac{\partial V}{\partial z} = \frac{k^2}{\kappa_r^2} \left( \frac{(1-\mu)Z}{\Delta_1^3} + \frac{\mu Z}{\Delta_2^3} \right), \quad (10)$$

$$\Delta_1 = \sqrt{(X+\mu)^2 + Y^2 + Z^2} = \frac{\delta_1}{\kappa_r}, \quad (11)$$

$$\Delta_2 = \sqrt{(X-1+\mu)^2 + Y^2 + Z^2} = \frac{\delta_2}{\kappa_r}. \quad (12)$$

Substituting

$$\frac{d\kappa_x}{dt} = -k \frac{\sqrt{\alpha(1+\beta^2)} \sin \theta}{\alpha - \beta \cos \theta} = -\frac{k}{\sqrt{(1+e)q}} \frac{\kappa_y}{\kappa_r}, \quad (13)$$

$$\frac{d\kappa_y}{dt} = k \sqrt{(1+e)q} \frac{(\alpha\beta + \cos \theta)}{\alpha - \beta \cos \theta} = \frac{k \sqrt{(1+e)q}}{e} \left( \frac{1}{\kappa_r} - \frac{1}{a} \right), \quad (14)$$

$$\frac{d\kappa_r}{dt} = k \sqrt{(1+e)q} \frac{\sin \theta}{(\alpha - \beta \cos \theta)} = \frac{ke}{\sqrt{(1+e)q}} \frac{\kappa_y}{\kappa_r}, \quad (15)$$

$$\kappa_x \frac{d\kappa_x}{dt} + \kappa_y \frac{d\kappa_y}{dt} = \kappa_r \frac{d\kappa_r}{dt}, \quad (16)$$

$$\kappa_x \frac{d\kappa_y}{dt} - \kappa_y \frac{d\kappa_x}{dt} = k \sqrt{(1+e)q}, \quad (17)$$

$$\frac{d^2 \kappa_x}{dt^2} = -k^2 \frac{\kappa_x}{\kappa_r^3}, \quad (18)$$

$$\frac{d^2 \kappa_y}{dt^2} = -k^2 \frac{\kappa_y}{\kappa_r^3}, \quad (19)$$

$$\frac{d^2 \kappa_r}{dt^2} = -k^2 \frac{1}{\kappa_r^2}, \quad (20)$$

then we have

$$\left\{ \begin{array}{l} \frac{d^2 X}{dt^2} + \frac{2}{\kappa_r} \frac{d\kappa_r}{dt} \frac{dX}{dt} - \frac{k^2}{\kappa_r^3} X - 2k \frac{\sqrt{(1+e)q}}{\kappa_r^2} \frac{dY}{dt} = -\frac{k^2}{\kappa_r^3} \left( \frac{(1-\mu)(X+\mu)}{\Delta_1^3} + \frac{\mu(X-1+\mu)}{\Delta_2^3} \right) \\ \frac{d^2 Y}{dt^2} + \frac{2}{\kappa_r} \frac{d\kappa_r}{dt} \frac{dY}{dt} - \frac{k^2}{\kappa_r^3} Y + 2k \frac{\sqrt{(1+e)q}}{\kappa_r^2} \frac{dX}{dt} = -\frac{k^2}{\kappa_r^3} \left( \frac{1-\mu}{\Delta_1^3} + \frac{\mu}{\Delta_2^3} \right) Y \\ \frac{d^2 Z}{dt^2} + \frac{2}{\kappa_r} \frac{d\kappa_r}{dt} \frac{dZ}{dt} - \frac{k^2}{\kappa_r^3} Z = -\frac{k^2}{\kappa_r^3} \left( \frac{1-\mu}{\Delta_1^3} + \frac{\mu}{\Delta_2^3} \right) Z \end{array} \right. \quad (21)$$

Converting  $t \rightarrow \theta$ ,

$$\begin{aligned} \frac{d}{dt} &= \frac{d\theta}{dt} \frac{d}{d\theta} = \frac{k}{\sqrt{\alpha(1+\beta^2)}} \frac{(1+\alpha\beta\cos\theta)^2}{\alpha-\beta\cos\theta} \frac{d}{d\theta} \\ &= \frac{k(1+\alpha^2)}{\sqrt{\alpha(1+\beta^2)}} \frac{1}{(1+\alpha\kappa_r)\kappa_r} \frac{d}{d\theta}, \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{d^2}{dt^2} &= \frac{k^2}{\alpha(1+\beta^2)} \frac{(1+\alpha\beta\cos\theta)^4}{(\alpha-\beta\cos\theta)^2} \left( \frac{d^2}{d\theta^2} - \frac{\beta\sin\theta}{1+\alpha\beta\cos\theta} \left( \frac{1+\alpha^2}{\alpha-\beta\cos\theta} + \alpha \right) \frac{d}{d\theta} \right) \\ &= \frac{k^2(1+\alpha^2)^2}{\alpha(1+\beta^2)} \frac{1}{(1+\alpha\kappa_r)^2 \kappa_r^2} \left( \frac{d^2}{d\theta^2} - \frac{\beta\kappa_y}{\sqrt{\alpha^2-\beta^2}} \left( \frac{1}{\kappa_r} + 2\alpha \right) \frac{d}{d\theta} \right). \end{aligned} \quad (23)$$

Hence we have

$$\left\{ \begin{array}{l} \frac{1+\alpha\beta\cos\theta}{\alpha(1+\beta^2)} \left( (\alpha-\beta\cos\theta) \frac{d^2 X}{d\theta^2} + \beta\sin\theta \frac{dX}{d\theta} - 2\sqrt{\alpha^2-\beta^2} \frac{dY}{d\theta} \right) \\ \quad \quad \quad = X - \frac{(1-\mu)(X+\mu)}{\Delta_1^3} - \frac{\mu(X-1+\mu)}{\Delta_2^3} \\ \frac{1+\alpha\beta\cos\theta}{\alpha(1+\beta^2)} \left( (\alpha-\beta\cos\theta) \frac{d^2 Y}{d\theta^2} + \beta\sin\theta \frac{dY}{d\theta} + 2\sqrt{\alpha^2-\beta^2} \frac{dX}{d\theta} \right) \\ \quad \quad \quad = Y - \frac{(1-\mu)Y}{\Delta_1^3} - \frac{\mu Y}{\Delta_2^3} \\ \frac{1+\alpha\beta\cos\theta}{\alpha(1+\beta^2)} \left( (\alpha-\beta\cos\theta) \frac{d^2 Z}{d\theta^2} + \beta\sin\theta \frac{dZ}{d\theta} \right) = Z - \frac{(1-\mu)Z}{\Delta_1^3} - \frac{\mu Z}{\Delta_2^3} \end{array} \right. \quad (24)$$

Integrating Eq.24 by  $X$ ,  $Y$ , and  $Z$ , respectively, and adding, we have

$$\begin{aligned}
& \int \frac{(1 + \alpha\beta \cos\theta)(\alpha - \beta \cos\theta)}{\alpha(1 + \beta^2)} \left\{ \frac{dX}{d\theta} d\left(\frac{dX}{d\theta}\right) + \frac{dY}{d\theta} d\left(\frac{dY}{d\theta}\right) + \frac{dZ}{d\theta} d\left(\frac{dZ}{d\theta}\right) \right\} \\
& + \int \frac{\beta \sin\theta(1 + \alpha\beta \cos\theta)}{\alpha(1 + \beta^2)} \left\{ \left(\frac{dX}{d\theta}\right) dX + \left(\frac{dY}{d\theta}\right) dY + \left(\frac{dZ}{d\theta}\right) dZ \right\} \\
& = \int (X dX + Y dY + Z dZ) + \frac{1 - \mu}{\Delta_1} + \frac{\mu}{\Delta_2} + C.
\end{aligned} \tag{25}$$

By partial integral for the left side, we have

$$\begin{aligned}
& \frac{(1 + \alpha\beta \cos\theta)(\alpha - \beta \cos\theta)}{2\alpha(1 + \beta^2)} \left\{ \left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2 + \left(\frac{dZ}{d\theta}\right)^2 \right\} \\
& + \frac{e}{2} \int \left\{ \left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2 + \left(\frac{dZ}{d\theta}\right)^2 \right\} \sin\theta d\theta \\
& - \left( \frac{1 - \mu}{\Delta_1} + \frac{\mu}{\Delta_2} \right) = \frac{X^2 + Y^2 + Z^2}{2} + C.
\end{aligned} \tag{26}$$

Converting  $\theta \rightarrow t$ , we have

$$\begin{aligned}
& \frac{\kappa_r^3}{2k^2} \left\{ \left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 + \left(\frac{dZ}{dt}\right)^2 \right\} \\
& + \frac{ke}{2\sqrt{(1+e)q}} \int \kappa_y \kappa_r \left\{ \left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 + \left(\frac{dZ}{dt}\right)^2 \right\} dt \\
& - \left( \frac{1 - \mu}{\Delta_1} + \frac{\mu}{\Delta_2} \right) = \frac{X^2 + Y^2 + Z^2}{2} + C
\end{aligned} \tag{27}$$

If  $e > 0$ , converting from  $t$  to the two-body coordinate  $\kappa_r$ , we have

$$\begin{aligned}
& \frac{e\kappa_y^2 \kappa_r}{2(1+e)q} \left\{ \left(\frac{dX}{d\kappa_r}\right)^2 + \left(\frac{dY}{d\kappa_r}\right)^2 + \left(\frac{dZ}{d\kappa_r}\right)^2 \right\} \\
& + \frac{k^2 e^2}{2(1+e)q} \int \kappa_y^2 \left\{ \left(\frac{dX}{d\kappa_r}\right)^2 + \left(\frac{dY}{d\kappa_r}\right)^2 + \left(\frac{dZ}{d\kappa_r}\right)^2 \right\} d\kappa_r \\
& - \left( \frac{1 - \mu}{\Delta_1} + \frac{\mu}{\Delta_2} \right) = \frac{X^2 + Y^2 + Z^2}{2} + C
\end{aligned} \tag{28}$$

Further converting Eq.(27) from  $(X, Y, Z) \rightarrow (x, y, z)$ , we have

$$\begin{aligned}
& \frac{1}{2} \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} \\
& + \frac{1}{2\kappa_r} \int \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} \left( \frac{d\kappa_r}{dt} \right) dt \\
& - k^2 \left( \frac{1-\mu}{\delta_1} + \frac{\mu}{\delta_2} \right) = \frac{k^2(x^2 + y^2 + z^2)}{2\kappa_r^3} + \frac{k^2 C}{\kappa_r}
\end{aligned} \tag{29}$$

Again converting  $t \rightarrow \theta$ , we have

$$\begin{aligned}
& \frac{(1 + \alpha\beta \cos\theta)^2}{2\alpha(1 + \beta^2)\kappa_r^2} \left\{ \left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 + \left( \frac{dz}{d\theta} \right)^2 \right\} \\
& + \frac{e}{2\kappa_r} \int \left\{ \left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 + \left( \frac{dz}{d\theta} \right)^2 \right\} \left( \frac{\sin\theta}{\kappa_r^2} \right) d\theta \\
& - \left( \frac{1-\mu}{\delta_1} + \frac{\mu}{\delta_2} \right) = \frac{x^2 + y^2 + z^2}{2\kappa_r^3} + \frac{C}{\kappa_r}
\end{aligned} \tag{30}$$

Further converting from  $\theta$  to two-body coordinate  $\kappa_r$ , we have

$$T + \frac{1}{\kappa_r} \int T d\kappa_r + V = \frac{k^2 r^2}{2\kappa_r^3} + \frac{k^2 C}{\kappa_r} \tag{31}$$

where

$$T = k^2 \left( \frac{1}{\kappa_r} - \frac{1}{2a} - \frac{(1+e)q}{2\kappa_r^2} \right) \left\{ \left( \frac{dx}{d\kappa_r} \right)^2 + \left( \frac{dy}{d\kappa_r} \right)^2 + \left( \frac{dz}{d\kappa_r} \right)^2 \right\} \tag{32}$$

and  $r^2 = x^2 + y^2 + z^2$ .

In Eqs.(26)-(31), the first term of the left side is the kinetic energy, and the third term is the potential energy. The second term of the left side is vanished in the case of  $e = 0$ , namely circular orbit. So the term is named ‘‘eccentric kinetic integral’’.

In the case of circular orbit,  $\kappa_r = a$ . Hence it is just identical to classical Jacobi integral,

$$\frac{a^3}{2k^2} \left\{ \left( \frac{dX}{dt} \right)^2 + \left( \frac{dY}{dt} \right)^2 + \left( \frac{dZ}{dt} \right)^2 \right\} - \left( \frac{1-\mu}{\Delta_1} + \frac{\mu}{\Delta_2} \right) = \frac{X^2 + Y^2 + Z^2}{2} + C. \tag{33}$$

or

$$\frac{1}{2} \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} - k^2 \left( \frac{1-\mu}{\delta_1} + \frac{\mu}{\delta_2} \right) = \frac{k^2(x^2 + y^2 + z^2)}{2a^3} + \frac{k^2 C}{a}. \tag{34}$$

where  $a$  is semimajor axis. Therefore the eccentric kinetic integral is an additional term of the generalized Jacobi integral.

In one-dimensional case, the generalized Jacobi integral is reduced to

$$\left(\frac{1}{\kappa_r} - \frac{1}{2a}\right) \left(\frac{dx}{d\kappa_r}\right)^2 + \frac{1}{\kappa_r} \int \left(\frac{1}{\kappa_r} - \frac{1}{2a}\right) \left(\frac{dx}{d\kappa_r}\right)^2 d\kappa_r - \left(\frac{1-\mu}{\delta_1} + \frac{\mu}{\delta_2}\right) = \frac{x^2}{2\kappa_r^3} + \frac{C}{\kappa_r}. \quad (35)$$

In this case, the problem can be solved if this non-linear first-order differential equation is solved.

### 3. Conclusion

Generalized Jacobi integral of the general restricted three-body problem has been derived using the projective anomaly. The simple expression is Eq.(31). The derivation is valid for all kind of Keplerian orbit type, circular, elliptical, parabolic, hyperbolic, and even two-body collisional linear orbit. The generalized Jacobi integral has an additional term to classical Jacobi integral. The term is named “eccentric kinetic integral” because the term is vanished in the case of circular orbit. In one-dimensional case, the problem can be solved if the generalized Jacobi integral is solved by any method.

### References

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